

THE HOPF MONOID OF COLORING PROBLEMS

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ABSTRACT. We study coloring problems, which are induced subposets P of a Boolean lattice, paired with an order ideal I from the poset of intervals, ordered by inclusion. We study a quasisymmetric function associated to coloring problems, called the chromatic quasisymmetric function, generalizing Stanley's chromatic symmetric function of a graph. We show that the chromatic quasisymmetric function is an Ehrhart quasisymmetric function, and that a transformation of the chromatic quasisymmetric function gives a Hilbert polynomial. Finally, we introduce combinatorial Hopf monoids in pointed set species, and show that coloring problems form the terminal object in the category of combinatorial Hopf monoids.

1. INTRODUCTION

In their landmark paper, Aguiar, Bergeron and Sottile [1] proved that quasisymmetric functions form the terminal combinatorial Hopf algebra, and used this to derive the generalized Dehn-Somerville relations. They showed how many well-known examples of quasisymmetric generating functions came from combinatorial Hopf algebras that had very simple characters. However, many examples of combinatorial Hopf algebras have more structure: several of them could be realized as being constructed from a Hopf monoid in Joyal's [8] category of combinatorial species, as was shown by Aguiar and Mahajan [2]. That is, graphs, posets, and matroids give rise to Hopf monoids in species, and the corresponding combinatorial Hopf algebras are obtained by applying a Fock functor.

Many of the examples of combinatorial Hopf algebras involve 'simple' characters φ that only take values 0 and 1 on a distinguished basis. Moreover, at the level of species, the multiplication sends pairs of basis elements to basis elements, and the basis elements h for which $\varphi(h) = 1$ also contain a lot of structure. To this end, we introduce combinatorial Hopf monoids in pointed set species. The intuition is that we have a collection of labeled combinatorial objects, and rules for combining and decomposing the structures. Moreover, we have a subcollection of 'stable' objects. We also describe how to obtain a combinatorial Hopf algebra from a combinatorial

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Hopf monoid. Moreover, the resulting quasisymmetric function invariants have an interpretation in terms of counting functions we call colorings.

We describe the terminal object in the category of combinatorial Hopf monoids, called the Hopf monoid of coloring problems. We also show that the resulting quasisymmetric functions are Ehrhart quasisymmetric functions, and their polynomial specializations are Hilbert polynomials of *relative* simplicial complexes. This generalizes work of Steingrímsson [10] from chromatic polynomials of graphs to arbitrary quasisymmetric functions coming from combinatorial Hopf monoids. It also generalizes some known results about poset polytopes, matroid polytopes, and generalized permutohedra.

The paper is organized as follows: first we define coloring problems, their chromatic polynomials and chromatic quasisymmetric functions. This part is elementary, and self-contained. Then we discuss the relationship between coloring problems and relative simplicial complexes. The motivation is that many quasisymmetric functions for combinatorial Hopf algebras also arise in studying Hilbert polynomials and Ehrhart polynomials, and we give one explanation for this phenomenon. Then we move into category theory, and discuss the category of pointed set species. We introduce the notion of combinatorial Hopf monoid, with many examples. The examples help to illustrate the abundance of combinatorial Hopf monoids already appearing in the literature, and thus helps us to emphasize that each of these other combinatorial objects are really coloring problems. Finally, we show that coloring problems form the terminal combinatorial Hopf monoid. As a corollary, our theorem on Hilbert polynomials is true for many numerical polynomial invariants.

2. COLORING PROBLEMS

In this section, we introduce the main character in this story: the coloring problem. The name comes from the fact that it generalizes graph coloring and hypergraph coloring.

Definition 1. Given a poset \mathbf{p} on a finite set N , let $\text{Int}(\mathbf{p})$ be the poset of intervals, ordered by inclusion. Then a coloring problem is a pair (\mathbf{p}, \mathbf{I}) such that:

- (1) \mathbf{p} is a collection of subsets of N , ordered by inclusion.
- (2) $\emptyset, N \in \mathbf{p}$.
- (3) \mathbf{I} is an order ideal of $\text{Int}(\mathbf{p})$.
- (4) For every $S \subseteq N$, $[S, S] \in \mathbf{I}$.

Now we define a coloring of a coloring problem. A *proper coloring* is a function $f : N \rightarrow \mathbb{N}$ such that, for every $n \in \mathbb{N}$, $[f^{-1}([n]), f^{-1}([n+1])] \in \mathbf{I}$. In other words, the subset $S \subset V$ of vertices colored $1, \dots, i$ must be an set in \mathbf{p} , and the subset T of vertices colored $i+1$ has to satisfy the condition that $[S, S \cup T] \in \mathbf{I}$.

First, we discuss how coloring problems generalize graph coloring, P -partitions, and M -generic functions of a matroid. This is all a special case

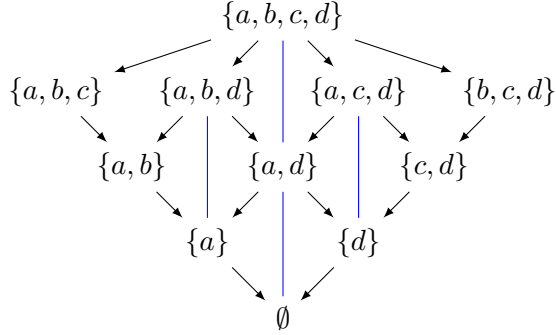


FIGURE 1. A coloring problem whose maximum stable intervals are indicated by the blue lines.

of Theorem below, where we show that coloring problems form part of a terminal object in the category of pairs of combinatorial Hopf monoids, via a map φ .

Given a graph \mathbf{g} , let $\varphi(\mathbf{g})$, be the coloring problem $(2^V, \mathbf{I})$, where \mathbf{I} consists of pairs $S \subseteq T$ where $T - S$ is an independent set. We see that a solution f to this coloring problem has the feature that $f^{-1}(i)$ is an independent set for all i . In other words, f is a proper coloring of the graph, and all such proper colorings are solutions to the coloring problem.

For a poset \mathbf{p} , the associated coloring problem is $(J(\mathbf{p}), \mathbf{I}(\mathbf{p}))$ where $J(\mathbf{p})$ are the order ideals of \mathbf{p} , ordered by inclusion, and $\mathbf{I}(\mathbf{p})$ consists of pairs $I \subseteq J$ of order ideals such that $J - I$ is an antichain. In this case, a solution to the coloring problem is a strict \mathbf{p}^* -partition, and all strict \mathbf{p}^* -partitions arise in this way.

For a matroid \mathbf{m} , the associated coloring problem is $(2^V, \mathbf{I})$, where \mathbf{I} consists of pairs $S \subseteq T$ such that $\mathbf{m}|_T/S$ has a unique basis. In this example, a solution to the coloring problem is a function $f : V \rightarrow \mathbb{N}$ which is minimized by a unique basis. These are called M -generic functions [3].

Finally, we define the chromatic quasisymmetric function, and chromatic polynomial of a coloring problem. Let $(x_i)_{i=1}^\infty$ be a sequence of commuting indeterminates.

Definition 2. Given a coloring problem (P, I) on node set N , the chromatic quasisymmetric function is defined by

$$\hat{\chi}(P, I) = \sum_f \prod_{n \in N} x_{f(n)}$$

where we sum over all proper colorings of (P, I) . Likewise, the chromatic polynomial is defined by $\chi(P, I, n) = \#$ of colorings $f : N \rightarrow \{1, \dots, n\}$. This is a polynomial function of n .

Let us prove some basic properties for the chromatic polynomial. We will give elementary proofs, although one could also prove these results using the

theory of Hopf monoids. First, chromatic polynomials and order polynomials are known to be multiplicative. We generalize this result, by defining products of coloring problems.

Definition 3. Let $\mathbf{c} = (\mathbf{p}, I)$ be a coloring problem on node set M , and $\mathbf{d} = (\mathbf{q}, J)$ be a coloring problem on node set N , which is disjoint from M . Then their product $\mathbf{c} \cdot \mathbf{d}$ is a coloring problem on $M \cup N$, given by $\mathbf{c} \cdot \mathbf{d} = (\mathbf{p} \cdot \mathbf{q}, I \cdot J)$, where $\mathbf{p} \cdot \mathbf{q} = \{X \cup Y : X \in \mathbf{p}, Y \in \mathbf{q}\}$ and $I \cdot J = \{[X \cup Y, X' \cup Y'] : [X, Y] \in I, [X', Y'] \in J\}$.

Proposition 4. Let \mathbf{c} be a coloring problem on N , and \mathbf{d} be a coloring problem on N , disjoint from M . Then $\hat{\chi}(\mathbf{c} \cdot \mathbf{d}) = \hat{\chi}(\mathbf{c})\hat{\chi}(\mathbf{d})$. Moreover, $\chi(\mathbf{c}, \mathbf{d}, x) = \chi(\mathbf{c}, x)\chi(\mathbf{d}, y)$.

Proof. Let $\mathbf{c} = (\mathbf{p}, I)$, and $\mathbf{d} = (\mathbf{q}, J)$. Let $h : M \sqcup N \rightarrow \mathbb{N}$ and define $f : M \rightarrow \mathbb{N}$ and $g : N \rightarrow \mathbb{N}$ to be the restriction of h to M and N . Observe that $h^{-1}(i) \in \mathbf{p} \cdot \mathbf{q}$ if and only if $f^{-1}(i) \in \mathbf{p}$ and $g^{-1}(i) \in \mathbf{q}$. Likewise, $[h^{-1}(i), h^{-1}(i+1)]$ is stable if and only if $[f^{-1}(i), f^{-1}(i+1)] \in I$ and $[g^{-1}(i), g^{-1}(i+1)] \in J$.

Thus, we have defined a map from the set of proper colorings of $(\mathbf{p}, I) \cdot (\mathbf{q}, J)$ to the set of pairs of colorings of (\mathbf{p}, I) and (\mathbf{q}, J) . This map is clearly a bijection, and the result follows. \square

There is also a generalization of the binomial theorem for chromatic polynomials of graphs: that is, $\chi(\mathbf{g}, x+y)$ can be expressed as a sum of products of chromatic polynomials. This result also can be generalized, and the generalization involves restrictions and contractions.

Definition 5. Let $\mathbf{c} = (\mathbf{p}, I)$ be a coloring problem on N , and let $S \subseteq N$. The *restriction* $\mathbf{c}|_S$ is non-zero if and only if $S \in \mathbf{p}$, in which case $\mathbf{c}|_S = (\mathbf{p}|_S, I|_S)$, where $\mathbf{p}|_S = \{T \in \mathbf{p} : T \subseteq S\}$, and $I|_S = \{[X, Y] \in I : Y \subseteq S\}$. The *contraction* of \mathbf{c} is non-zero if and only if $S \in \mathbf{p}$, in which case $\mathbf{c}/S = (\mathbf{p}/S, I/S)$, where $\mathbf{p}/S = \{X \subset N - S : X \cup S \in \mathbf{p}\}$, and $I/S = \{[X, Y] : X \subset Y \subset N - S, [X \cup S, Y \cup S] \in I\}$.

Proposition 6. Let \mathbf{c} be a coloring problem on N , and let x and y be indeterminates. Then $\chi(\mathbf{c}, x+y) = \sum_{S \in P} \chi(\mathbf{c}|_S, x)\chi(\mathbf{c}/S, y)$.

Proof. Let $\mathbf{c} = (\mathbf{p}, I)$, x and y be positive integers, and let $h : N \rightarrow [x+y]$ be a proper coloring. Define $S = h^{-1}([x])$. Observe that $S \in \mathbf{p}$. Moreover, restricting h to S gives a proper coloring f of $(\mathbf{p}|_S, I|_S)$ using the colors from $[x]$. Likewise, define $g : N \setminus S$ by $g(v) = h(v) - x$. Then $[g^{-1}(i), g^{-1}(i+1)] \in I/S$ if and only if $[g^{-1}(i) \cup S, g^{-1}(i+1) \cup S] \in I$, which is true if and only if $[h^{-1}(i+x), h^{-1}(i+x+1)] \in I$ for all $i \in [y]$. Thus g is also a proper coloring of $\chi(\mathbf{p}/S, I/S, y)$. Thus, we have defined a function from the set of $(x+y)$ -colorings of (\mathbf{p}, I) to the set of triples (S, f, g) , where f is an x -coloring of $(\mathbf{p}|_S, I|_S)$, and g is a y -coloring of $(\mathbf{p}/S, I/S)$. Again this function is a bijection. \square

3. THE EHRHART-HILBERT CONNECTION

Steingrímsson [10] introduced the coloring ideal, and coloring complex of a graph. The coloring complex has been used [6, 7] to obtain new inequalities satisfied by the chromatic polynomial of a graph. It is an open question to find related results for other families of polynomials, and it is noted by Hersh and Swarts that their results do not extend for characteristic polynomials of matroids. In this section, we show that the chromatic polynomial of a coloring problem is always a Hilbert polynomial for a Stanley-Reisner module. We also show that the chromatic polynomial is always an Ehrhart polynomial. In some sense, this gives some explanation why chromatic polynomials of graphs, and order polynomials of posets, can be understood from the perspectives of Hopf algebras, Ehrhart theory, and combinatorial commutative algebra, by showing that chromatic polynomials of coloring problems arise in these three areas.

3.1. Relative Stanley-Reisner modules, Hilbert Polynomial. A *relative simplicial complex* is a pair (Γ, Δ) where $\Gamma \subseteq \Delta$, and Δ is a simplicial complex. Given Δ with vertices S , we let $\mathbb{C}[S]$ be the polynomial ring with indeterminates s_1, \dots, s_k , the vertices of S . The *Stanley-Reisner ideal* for Δ is generated by $\langle \sigma \subseteq S : \sigma \notin \Delta \rangle$, and the *Stanley-Reisner module* for (Γ, Δ) is I_Γ/I_Δ . The module is graded by total degree, and its Hilbert function $H(\Gamma, \Delta)(n)$ is the number of monomials of degree n in the module. It is known that the Hilbert function is in fact a polynomial: details can be found in [9].

Let $\mathbf{c} = (\mathbf{p}, I)$ be a coloring problem. Let $\Delta(\mathbf{p})$ be the order complex of \mathbf{p} . Note that we are *not* removing the top and bottom elements of \mathbf{p} , so the top and bottom elements are cone points in $\Delta(\mathbf{p})$. Let $\Gamma(\mathbf{p}, I)$ consist of those chains $S_1 \subseteq S_2 \subseteq \dots \subseteq S_k \subseteq N$, such that $[S_i, S_{i+1}] \notin I$ for some i . We call $(\Delta(\mathbf{p}), \Gamma(\mathbf{p}, I))$ the *relative order complex* of (\mathbf{p}, I) .

Theorem 7. *Let (\mathbf{p}, I) be a coloring problem on N , with relative order complex $(\Delta(\mathbf{p}), \Gamma(\mathbf{p}, I))$. Then $H(\Delta(\mathbf{p}), \Gamma(\mathbf{p}, I))(n) = \chi(\mathbf{p}, I)(n+1)$.*

Proof. Observe that $H(\Delta(\mathbf{p}), \Gamma(\mathbf{p}, I))(n+1)$ counts the number of monomials of the Stanley-Reisner module of degree $n+1$. After rearranging terms in the monomial, we are counting the number of monomials $S_1^{a_1} \dots S_k^{a_k}$ such that:

- (1) a_1, \dots, a_k are all positive, and $a_1 + a_2 + \dots + a_k = n$,
- (2) $S_1 \subset S_2 \subset \dots \subset S_k$,
- (3) $[S_i, S_{i+1}] \in I$ for all i .

However, if we replace $S_i^{a_i}$ with a_i copies of S_i , then we obtain a sequence $T_1 \subseteq T_2 \subseteq \dots \subseteq T_n$ such that $[T_i, T_{i+1}] \in I$ for all i . However, such a sequence defines a proper coloring $f : N \rightarrow [n+1]$. For each $i \in [n]$, define $f^{-1}(i) = T_i \setminus T_{i-1}$, and $f^{-1}(n+1) = N \setminus T_n$. Likewise, given a proper coloring f , we can study the color classes, and obtain a multichain, and thus a monomial as above. \square

An alternate proof follows from work of Breuer and Klivans [5], using generating functions, and scheduling problems. We remark that coloring problems are scheduling problems. However, for scheduling problems they do not have a Stanley-Reisner ideal, nor a relative simplicial complex.

3.2. Ehrhart Quasisymmetric Function. Now we discuss the relationship between our chromatic quasisymmetric functions and Ehrhart theory. The Coxeter complex of type A is the order complex on the boolean lattice $2^N \setminus N$. We let Σ_N denote the Coxeter complex of type A on the set N . Note here that, since we are not fixing a linear order on N , we are viewing \mathbb{R}^N to be the set of functions $f : N \rightarrow \mathbb{R}$. In particular, a lattice point in the positive orthant of \mathbb{R}^N is the same thing as a function $f : N \rightarrow \mathbb{N} \setminus \{0\}$.

Given a face $F = \emptyset \subset S_1 \subset S_2 \subset \cdots \subset S_m \subset N$ of Σ_N , there is a corresponding polyhedral cone in the positive orthant $\mathbb{R}_{\geq 0}^N$. The cone is given by the equations $x_i < x_j$ whenever $i \in S_k, j \notin S_k$ for some Δ , and $x_i = x_j$ whenever $i \in S_k$ if and only if $j \in S_k$. For example, for the flag $\{2, 4\} \subset \{1, 2, 4, 7\} \subset \{1, 2, 3, 4, 7, 9\}$, we obtain the polyhedral cone given by $x_2 = x_4 < x_1 = x_7 < x_3 = x_9 < x_5 = x_8$. We call this the geometric realization of F , and denote it by $|F|$. Given a lattice point $\mathbf{a} \in \mathbb{R}_{> 0}^I$, we let $\mathbf{x}_{\mathbf{a}} = \prod_{i \in I} x_{a_i}$ be its monomial, where the coordinates of \mathbf{a} are encoded in the *indices*, not the exponents. Given a collection of cones C , the *Ehrhart quasisymmetric function* for C is given by

$$E_C = \sum_{\mathbf{a}} \mathbf{x}_{\mathbf{a}}$$

where the sum is over all lattice points which lie in some cone of C . Since $E_C = \sum_{F \in C} M_{\text{type}(F)}$, this is a quasisymmetric function, first appearing in the work of [4]. Finally, define $|\Delta(\mathbf{p}) \setminus \Gamma(\mathbf{p}, I)| = \bigcup_{F \in \Delta(\mathbf{p}) \setminus \Gamma(\mathbf{p}, I)} |F|$.

Theorem 8. *Let (\mathbf{p}, I) be a coloring problem on N , and let $(\Delta(\mathbf{p}), \Gamma(\mathbf{p}, I))$ be the relative order complex. Then*

$$E_{C(\Delta(\mathbf{p}) \setminus \Gamma(\mathbf{p}, I))} = \hat{\chi}(\mathbf{p}, I)$$

Proof. We associate, to each proper coloring $f : N \rightarrow \mathbb{N}$, a flag $F \in \Delta(\mathbf{p}) \setminus \Gamma(\mathbf{p}, I)$. Consider the infinite sequence $f^{-1}(1), f^{-1}(2), \dots$, of subsets of N . There are only finitely many distinct entries in the sequence, which we write in order as $F = S_1 \subset S_2 \subset \cdots \subset S_k$. Moreover, $F \in \Delta(\mathbf{p}) \setminus \Gamma(\mathbf{p}, I)$. We call this the flag associated with f . Then we observe that f is an interior lattice point of $|F|$.

Likewise, given an interior lattice point f in a cone $|F|$, we see that f is a proper coloring. Moreover, the flag associated to f is F . Thus, we have a bijection between proper colorings, and interior lattice points. \square

4. COMBINATORIAL HOPF MONOIDS

In this section, we study Hopf monoids in the category of pointed set species. The motivation is that many examples of combinatorial Hopf algebras come from Hopf monoids in linear species. Oftentimes the linear Hopf monoid has a distinct basis, such that all of the algebraic operations only involve the basis elements and the zero vector. It turns out that these linear Hopf monoids can be understood as ‘linearizing’ combinatorial Hopf monoids in pointed set species. Moreover, there is some advantage to this approach: we can show that the terminal combinatorial Hopf monoid is the Hopf monoid of coloring problems.

First, we introduce pointed set species. A *pointed set* consists of a pair (x, S) where S is a set, and $x \in S$, called the base point. A morphism of pointed sets $f : (x, S) \rightarrow (y, T)$ consists of a function $f : S \rightarrow T$ such that $f(x) = y$. Informally, we view the base point as being the analogue of the zero vector in a vector space.

Given two pointed sets $(x, S), (y, T)$, their *wedge sum* $S \vee T$ is given by $(S \setminus \{x\}) \sqcup T$, with y as the base point. The *smash product* $S \wedge T$ is the quotient of $S \times T$ by the equivalence relation $(s, t) \simeq (s', t')$ if $s = s' = x$ or $t = t' = y$. The base point for the smash product is the equivalence class of (x, y) .

A *pointed set species* is a functor $F : \text{Set} \rightarrow P\text{Set}$ from the category of sets with bijections, to the category of pointed sets with morphisms. From now on, we abuse notation and let 0 to denote the base point of F_N , for any finite set N . Here we list several examples of pointed set species. In each case, we describe the non-zero elements only.

G Given a finite set N , let G_N denote the collection of graphs with vertex set N . Given a bijection $\sigma : M \rightarrow N$, and a graph $\mathbf{g} \in G_M$, define $G_\sigma(\mathbf{g})$ to be the graph on N with edges ij if and only if $\sigma^{-1}(i)\sigma^{-1}(j)$ is an edge of \mathbf{g} . Then this gives rise to a pointed set species G , the species of graphs.

HG Given a finite set N , let HG_N denote the collection of hypergraphs with vertex set N . Given a bijection $\sigma : M \rightarrow N$, and a hypergraph $\mathbf{h} \in HG_M$, define $HG_\sigma(\mathbf{h})$ to be the hypergraph on N with edges $S \subseteq N$ if and only if $\sigma^{-1}(S)$ is an edge of \mathbf{h} . Then this gives rise to a pointed set species G , the species of hypergraphs.

P Given a finite set N , let P_N denote all partial orders on N . Given a bijection $\sigma : M \rightarrow N$, and a partial order $\mathbf{p} \in P_M$, define $P_\sigma(\mathbf{p})$ to be the partial order on N given by $x \leq y$ if and only if $\sigma^{-1}(x) \leq_{\mathbf{p}} \sigma^{-1}(y)$. Then this gives rise to a pointed set species P , the species of posets.

M Given a finite set N , let M_N denote the collection of matroids with ground set N . Given a bijection $\sigma : M \rightarrow N$, and a matroid $\mathbf{m} \in M_M$, define $M_\sigma(\mathbf{m})$ to be the matroid on N where a set S is a basis if and

only if $\sigma^{-1}(S)$ is a basis of \mathbf{m} . Then this gives rise to a pointed set species \mathbf{M} , the species of matroids.

A Given a finite set N , let \mathbf{A}_N denote the collection of antimatroids with ground set N . Given a bijection $\sigma : M \rightarrow N$, and an antimatroid $\mathbf{a} \in \mathbf{A}_M$, define $\mathbf{A}_\sigma(\mathbf{a})$ to be the antimatroid on N where a set S is feasible if and only if $\sigma^{-1}(S)$ is a feasible set of \mathbf{a} . Then this gives rise to a pointed set species \mathbf{A} , the species of antimatroids.

Gr Given a finite set N , let \mathbf{Gr}_N denote the collection of greedoids with vertex set N . Given a bijection $\sigma : M \rightarrow N$, and a greedoid $\mathbf{g} \in \mathbf{Gr}_M$, define $\mathbf{Gr}_\sigma(\mathbf{g})$ to be the greedoid on N where a set S is feasible if and only if $\sigma^{-1}(S)$ is a feasible set of \mathbf{g} . Then this gives rise to a pointed set species \mathbf{Gr} , the species of greedoids.

C Given a finite set N , let \mathbf{C}_N denote the collection of coloring problems on N . Given a bijection $\sigma : M \rightarrow N$, and a coloring problem (\mathbf{p}, I) on M , we define $\sigma(\mathbf{p}, I)$ to be the coloring problem (\mathbf{q}, J) , where $\mathbf{q} = \{\sigma(S) : S \in \mathbf{p}\}$, and $J = \{[\sigma(S), \sigma(T)] : [S, T] \in I\}$. This forms a species, the species of coloring problems.

Given two pointed set species \mathbf{F}, \mathbf{G} , we define the *wedge sum* by:

$$(\mathbf{F} + \mathbf{G})_N = \mathbf{F}_N \vee \mathbf{G}_N$$

and their *Cauchy product* by:

$$(\mathbf{F} \cdot \mathbf{G})_N = \bigvee_{N=S \sqcup T} \mathbf{F}_S \wedge \mathbf{G}_T$$

The unit species $\mathbf{1}$ is given by $\mathbf{1}_\emptyset = \{0, 1\}$, and $\mathbf{1}_S = \{0\}$ for all $S \neq \emptyset$.

We state the following theorem without proof. The definition of symmetric monoidal category, complete with all the necessary commutative diagrams, can be found in [2].

Theorem 9. *The category of pointed set species is a symmetric monoidal category with respect to Cauchy product.*

4.1. Bimonoid in pointed set species. A pointed set species is *connected* if $|\mathbf{F}_\emptyset| = 2$. All pointed set species will be assumed to be connected, and we assume $\mathbf{F}_\emptyset = \{0, 1\}$, with 0 as base point. For the purposes of this paper, a Hopf monoid is thus a bimonoid object in the category of pointed set species, with respect to the Cauchy product. For the sake of completeness, we now proceed to give an elementary definition. First, given a bijection $\sigma : N \rightarrow M$, and $S \subseteq N$, we let $\sigma|_S : S \rightarrow \sigma(S)$ denote the restriction of σ to S .

4.1.1. Monoids. A species \mathbf{F} is a *monoid* if, for every finite set decomposition $N = S \sqcup T$, and elements $\mathbf{x} \in \mathbf{F}_S, \mathbf{y} \in \mathbf{F}_T$, there is a product $\mathbf{x} \cdot \mathbf{y} \in \mathbf{F}_N$, subject to:

- (1) naturality: given a bijection $\sigma : N \rightarrow M$, we have $\mathbf{F}_\sigma(\mathbf{x} \cdot \mathbf{y}) = \mathbf{F}_{\sigma|_S}(\mathbf{x}) \cdot \mathbf{F}_{\sigma|_T}(\mathbf{y})$.

- (2) associativity: given another finite set U , and $\mathbf{z} \in \mathbf{F}_U$, we have $(\mathbf{x} \cdot \mathbf{y}) \cdot \mathbf{z} = \mathbf{x} \cdot (\mathbf{y} \cdot \mathbf{z})$.
- (3) unital: $1 \cdot \mathbf{x} = \mathbf{x} = \mathbf{x} \cdot 1$.
- (4) absorption: if $\mathbf{x} = 0$ or $\mathbf{y} = 0$, then $\mathbf{x} \cdot \mathbf{y} = 0$.

Remark 10. Give disjoint finite sets S, T such that $S \sqcup T = N$, the operation $(\mathbf{x}, \mathbf{y}) \rightarrow \mathbf{x} \cdot \mathbf{y}$ gives rise to a function $\mu_{S,T} : \mathbf{F}_S \times \mathbf{F}_T \rightarrow \mathbf{F}_N$. Moreover, due to absorption, we get a well-defined map $\mu_{S,T} : \mathbf{F}_S \wedge \mathbf{F}_T \rightarrow \mathbf{F}_N$. If we take the wedge sum of these maps, over all S, T , we obtain a function $\mu : \mathbf{F} \cdot \mathbf{F} \rightarrow \mathbf{F}$. Naturality ensures that μ is a natural transformation. The axioms of associativity and unitality ensure that (\mathbf{F}, μ, ι) is a monoid object in the sense of Aguiar and Mahajan.

Here are some examples of monoids.

- G** The species of graphs forms a monoid. The product is given by $\mathbf{g} \cdot \mathbf{h} = \mathbf{g} \sqcup \mathbf{h}$, the disjoint union of graphs.
- HG** The species of hypergraphs forms a monoid. The product is given by $\mathbf{g} \cdot \mathbf{h} = \mathbf{g} \sqcup \mathbf{h}$, the disjoint union of hypergraphs.
- P** The species of partial orders forms a monoid. The product is given by $\mathbf{p} \cdot \mathbf{q} = \mathbf{p} \sqcup \mathbf{q}$, the disjoint union of partial orders.
- M** Matroids form a monoid under the direct sum operation.
- A** Antimatroids form a monoid under direct sum.
- Gr** Greedoids form a monoid under direct sum.

Theorem 11. *The species \mathcal{C} of coloring problems is a monoid.*

Proof. Given three coloring problems (\mathbf{p}, I) , (\mathbf{q}, J) , and (\mathbf{r}, K) , we see that $\mathbf{p} \cdot (\mathbf{q} \cdot \mathbf{r}) = \{X \sqcup Y \sqcup Z : X \in \mathbf{p}, Y \in \mathbf{q}, Z \in \mathbf{r}\} = (\mathbf{p} \cdot \mathbf{q}) \cdot \mathbf{r}$, and similarly $I \cdot (J \cdot K) = \{[X \sqcup Y \sqcup Z, X' \sqcup Y' \sqcup Z'] : [X, X'] \in I, [Y, Y'] \in J, [Z, Z'] \in K\} = (I \cdot J) \cdot K$. Thus, \mathcal{C} is a monoid. \square

4.1.2. Comonoids. A species \mathbf{F} is a *comonoid* if, for every finite set decomposition $N = S \sqcup T$, and element $\mathbf{x} \in \mathbf{F}_N$, there is a restriction $\mathbf{x}|_S \in \mathbf{F}_S$, and a contraction $\mathbf{x}/S \in \mathbf{F}_T$, subject to:

- (1) naturality: given a bijection $\sigma : N \rightarrow M$, we have $\mathbf{F}_\sigma(\mathbf{x})|_S = \mathbf{F}_{\sigma|_S}(\mathbf{x}|_S)$ and $\mathbf{F}_\sigma(\mathbf{x})/S = \mathbf{F}_{\sigma|_T}(\mathbf{x}/S)$.
- (2) coassociativity: given $R \subset S$, $\mathbf{x}|_S/R = (\mathbf{x}/R)|_{S \setminus R}$. Moreover, if $\mathbf{x}|_S/R \neq 0$, then $(\mathbf{x}|_S)|_R = \mathbf{x}|_R$, and $\mathbf{x}/S = (\mathbf{x}/R)/(S \setminus R)$.
- (3) counital: $\mathbf{x}|_N = \mathbf{x} = \mathbf{x}/\emptyset$.
- (4) zero conditions: $\mathbf{x}|_S = 0$ if and only if $\mathbf{x}/S = 0$. Moreover, $0|_S = 0$ and $0/S = 0$ for all $S \subseteq N$.

Remark 12. Give disjoint finite sets S, T such that $S \sqcup T = N$, the operation $\mathbf{x} \rightarrow (\mathbf{x}|_S, \mathbf{x}/S)$ gives rise to a function $\Delta_{S,T} : \mathbf{F}_N \rightarrow \mathbf{F}_S \wedge \mathbf{F}_T$. If we take the wedge sum of these maps, over all S, T , we obtain a function $\Delta : \mathbf{F} \rightarrow \mathbf{F} \cdot \mathbf{F}$. The axioms above ensure that $(\mathbf{F}, \Delta, \epsilon)$ is a comonoid object in the sense of Aguiar and Mahajan.

Here are some examples of comonoids.

- G** The species of graphs forms a comonoid. Given a graph \mathbf{g} , and $S \subseteq N$, $\mathbf{g}|_S$ is the induced subgraph on S , and \mathbf{g}/S is the induced subgraph on $N - S$.
- HG** The species of hypergraphs forms a comonoid. Given a hypergraph \mathbf{g} and $S \subseteq N$, $\mathbf{g}|_S$ is the induced subhypergraph on S , and \mathbf{g}/S is the induced subhypergraph on $N - S$.
- P** The species of partial orders forms a comonoid. Given a partial order \mathbf{p} , and $S \subseteq N$, we define $\mathbf{p}|_S = \mathbf{p}/S = 0$ if S is not an order ideal of \mathbf{p} . If S is an order ideal, then $\mathbf{p}|_S$ and \mathbf{p}/S are the induced subposets on S and $N - S$ respectively.
- M** Matroids form a comonoid: given a matroid \mathbf{m} , and $S \subset N$, we define $\mathbf{m}|_S$ to be the restriction, and \mathbf{m}/S to be the contraction of matroids.
- A** Antimatroids form a comonoid. Given an antimatroid \mathbf{a} , and $S \subseteq N$, we define $\mathbf{a}|_S = \mathbf{a}/S = 0$ if $S \notin \mathbf{a}$. Otherwise, we define $\mathbf{a}|_S$ and \mathbf{a}/S to be the usual restriction and contraction.
- Gr** Greedoids form a comonoid. Here restriction and contraction are only defined when S is rank feasible. A rank feasible set S is one where the rank of S is the maximum size of $S \cap B$, where B is a basis.

A *combinatorial comonoid* \mathbf{C} is a comonoid such that, for all $S \subset T \subset N$, and $\mathbf{x} \in \mathbf{C}_N$, if $\mathbf{x}|_S \neq 0$ and $\mathbf{x}|_T \neq 0$, then $\mathbf{x}|_T/S \neq 0$. We note that this definition does not have a previous analogue for combinatorial Hopf algebras; however, all of our examples are combinatorial comonoids.

Theorem 13. *The species \mathbf{C} of coloring problems is a combinatorial comonoid.*

Proof. We show that \mathbf{C} is a comonoid. Let (\mathbf{p}, I) be a coloring problem on N , and let $R \subseteq S \subseteq N$, such that $R, S \in \mathbf{p}$. Then clearly $\mathbf{p}|_S|_R = \mathbf{p}|_R$, and $I|_S|_R = I|_R$. Similarly, $(\mathbf{p}/R)/(S \setminus R) = \{X \subseteq N \setminus S : X \cup (S \setminus R) \in \mathbf{p}/R\} = \{X \subseteq N \setminus S : X \cup S \in \mathbf{p}\} = \mathbf{p}/S$. and $\mathbf{p}|_S/R = (\mathbf{p}/R)|_{S \setminus R}$.

We check that \mathbf{C} is a combinatorial comonoid: given a coloring problem (\mathbf{p}, I) , let $S \subset T \subset N$ be such that $\mathbf{p}|_T \neq 0$ and $\mathbf{p}/S \neq 0$. Then $S, T \in \mathbf{p}$. Hence $S \in \mathbf{p}|_T$, and thus $\mathbf{p}|_T/S \neq 0$. Similarly, $I|_T/S \neq 0$, and thus coloring problems form a combinatorial comonoid. \square

4.1.3. Bimonoids. . A species \mathbf{F} is a *bimonoid* if it is a monoid and a comonoid, and satisfies both of the following additional conditions:

- (1) For every two decompositions $N = S \sqcup T = S' \sqcup T'$, $\mathbf{x} \in \mathbf{F}_S$, $\mathbf{y} \in \mathbf{F}_T$, we have: $(\mathbf{x} \cdot \mathbf{y})|_{S'} = \mathbf{x}|_{S \cap S'} \cdot \mathbf{y}|_{T \cap S'}$, and $(\mathbf{x} \cdot \mathbf{y})/S' = \mathbf{x}/(S \cap S') \cdot \mathbf{y}/(T \cap S')$.
- (2) We also have $1 \neq 0$.

All of our listed examples of monoids and comonoids are in fact bimonoids.

Proposition 14. *If \mathbf{B} is a connected bimonoid, then for all $\mathbf{x} \in \mathbf{B}_N$ and all $\mathbf{y} \in \mathbf{B}_M$, we have $\mathbf{x} \cdot \mathbf{y} = 0$ if and only if $\mathbf{x} = 0$ or $\mathbf{y} = 0$.*

Proof. If $\mathbf{x} = 0$, or $\mathbf{y} = 0$, then $\mathbf{x} \cdot \mathbf{y} = 0$ follows from the definition of monoid. So suppose $\mathbf{x} \cdot \mathbf{y} = 0$. Then $(\mathbf{x} \cdot \mathbf{y})|_\emptyset = 0|_\emptyset = 0$. However, since \mathbf{B} is a bimonoid, $(\mathbf{x} \cdot \mathbf{y})|_\emptyset = \mathbf{x}|_\emptyset \cdot \mathbf{y}|_\emptyset$. If $\mathbf{x}|_\emptyset = \mathbf{y}|_\emptyset = 1$, then we have $1 \cdot 1 = 0$. However, by unitality, $1 \cdot 1 = 1 \neq 0$. Thus, either $\mathbf{x}|_\emptyset$ or $\mathbf{y}|_\emptyset$ is not one. Since \mathbf{B} , is connected, $\mathbf{x}|_\emptyset = 0$ or $\mathbf{y}|_\emptyset = 0$. Without loss of generality, suppose $\mathbf{x}|_\emptyset = 0$. Then $\mathbf{x} = \mathbf{x}/\emptyset = 0$. \square

4.2. Combinatorial Hopf monoid. Now we define a combinatorial Hopf monoid. The motivation is that many combinatorial Hopf algebras are studied, where the character φ only takes on the values 0 and 1, so φ is essentially defined by taking a subHopf algebra generated by a subset of the basis.

Given two species \mathbf{F} and \mathbf{G} , we say that $\mathbf{F} \subseteq \mathbf{G}$ is a *subspecies* if, for all N , $\mathbf{F}_N \subseteq \mathbf{G}_N$, and furthermore, for any bijection $\sigma : M \rightarrow N$, and $\mathbf{x} \in \mathbf{G}_M$, we have $\mathbf{F}_\sigma(\mathbf{x}) \in \mathbf{G}_N$.

A *submonoid* is a subspecies $\mathbf{M} \subset \mathbf{H}$ such that, for all $\mathbf{x} \in \mathbf{M}_S$, and $\mathbf{y} \in \mathbf{M}_T$, we have $\mathbf{x} \cdot \mathbf{y} \in \mathbf{M}_{S \sqcup T}$. A submonoid is a two-sided *ideal* if, whenever $\mathbf{x} \in \mathbf{H}_S$, $\mathbf{y} \in \mathbf{M}_T$, we have $\mathbf{x} \cdot \mathbf{y} \in \mathbf{M}_{S \sqcup T}$, and $\mathbf{y} \cdot \mathbf{x} \in \mathbf{M}_{S \sqcup T}$. A *subcomonoid* is a subspecies $\mathbf{C} \subset \mathbf{H}$ such that, for all $\mathbf{x} \in \mathbf{C}_N$, and $S \subseteq N$, $\mathbf{x}|_S \in \mathbf{C}_S$ and $\mathbf{x}/S \in \mathbf{C}_{N \setminus S}$. Finally, a *Hopf submonoid* is a subspecies \mathbf{C} that is both a submonoid and a subcomonoid.

A *combinatorial Hopf monoid* \mathbf{H} is a triple $(\mathbf{H}, \mathbf{S}, \mathbf{I})$ satisfying:

- (1) $\mathbf{H} = \mathbf{S} \wedge \mathbf{I}$, where \mathbf{S} is a Hopf submonoid of \mathbf{H} , and \mathbf{I} is an ideal of \mathbf{H} .
- (2) \mathbf{H} is a combinatorial comonoid.

When describing a combinatorial Hopf monoid, we refer to the nonzero elements of \mathbf{S} as the *stable* structures. In all of our examples, we describe what the stable structures are.

- \mathbf{G} The species of graphs is a combinatorial Hopf monoid, with stable structures given by edgeless graphs.
- \mathbf{HG} The species of hypergraphs is a combinatorial Hopf monoid, with stable structures given by edgeless hypergraphs.
- \mathbf{P} The species of partial orders is a combinatorial Hopf monoid, with stable structures given by antichain posets.
- \mathbf{M} Matroids form a combinatorial Hopf monoid, with stable structures given by split matroids, which are matroids where every element is a loop or coloop.
- \mathbf{A} Antimatroids form a is a combinatorial Hopf monoid, with stable structures given by boolean lattices.
- \mathbf{Gr} Greedoids is a combinatorial Hopf monoid, with stable structures given by split matroids.

We claim that \mathbf{C} is a combinatorial Hopf monoid, where stable structures are those coloring problems of the form $(\mathbf{p}, \text{Int}(\mathbf{p}))$.

Theorem 15. *The species \mathbf{C} of coloring problems is a combinatorial Hopf monoid.*

Proof. First, we show that \mathbf{C} is a bimonoid. We show that the restriction of the product is the product of the restrictions. First, for posets, we see that

$$\begin{aligned} (\mathbf{p} \cdot \mathbf{q})|_S &= \{X \cup Y \subseteq S : X \in \mathbf{p}, Y \in \mathbf{q}\} \\ &= \{X \cup Y : X \in \mathbf{p}, X \subseteq S \cap M, Y \in \mathbf{q}, Y \subseteq S \cap N\} \\ &= \{X \cup Y : X \in \mathbf{p}|_{S \cap M}, Y \in \mathbf{q}|_{S \cap N}\} \\ &= \mathbf{p}|_{S \cap M} \cdot \mathbf{q}|_{S \cap N}. \end{aligned}$$

For the interval ideals, we see that

$$\begin{aligned} (I \cdot J)|_S &= \{[X \cup Y, X' \cup Y'] : [X, X'] \in I, [Y, Y'] \in J, X' \cup Y' \subseteq S\} \\ &= \{[X \cup Y, X' \cup Y'] : [X, X'] \in I, X' \subseteq S \cap M, [Y, Y'] \in J, Y' \subseteq S \cap N\} \\ &= \{[X \cup Y, X' \cup Y'] : [X, X'] \in I|_{S \cap M}, [Y, Y'] \in J|_{S \cap N}\} \\ &= I|_{S \cap M} \cdot J|_{S \cap N}. \end{aligned}$$

Now we study the quotient of a product. For posets, we compute that

$$\begin{aligned} (\mathbf{p} \cdot \mathbf{q})/S &= \{X \cup Y \subseteq (M \cup N) \setminus S : X \cup Y \cup S \in \mathbf{p} \cdot \mathbf{q}\} \\ &= \{X \cup Y \subseteq (M \cup N) \setminus S : X \cup (S \cap M) \in \mathbf{p}, Y \cup (S \cap N) \in \mathbf{q}\} \\ &= \{X \cup Y \subseteq (M \cup N) \setminus S : X \in \mathbf{p}/S \cap M, Y \in \mathbf{q}/S \cap N\} \\ &= \mathbf{p}/S \cap M \cdot \mathbf{q}/S \cap N. \end{aligned}$$

Finally, for interval ideas, we see that

$$\begin{aligned} (I \cdot J)/S &= \{[X \cup Y, X' \cup Y'] : X' \cup Y' \subseteq (M \cup N) \setminus S, [X \cup Y \cup S, X' \cup Y' \cup S] \in I \cdot J\} \\ &= \{[X \cup Y, X' \cup Y'] : X' \cup Y' \subseteq (M \cup N) \setminus S, [X \cup (S \cap M), X' \cup (S \cap M)] \in I, \\ &\quad [Y \cup (S \cap N), Y' \cup (S \cap N)] \in J\} \\ &= \{[X \cup Y, X' \cup Y'] : X' \cup Y' \subseteq (M \cup N) \setminus S, [X, X'] \in I/(S \cap M), [Y, Y'] \in J/(S \cap N)\} \\ &= I/S \cap M \cdot J/S \cap N. \end{aligned}$$

To show that the stable coloring problems form a subcomonoid, it is enough to note that, for any poset \mathbf{p} on N , with $S \in \mathbf{p}$, we have $\text{Int}(\mathbf{p})|_S = \text{Int}(\mathbf{p}|_S)$ and $\text{Int}(\mathbf{p})/S = \text{Int}(\mathbf{p}/S)$. Thus the restriction and contraction of a stable coloring problem must be stable.

Suppose that (\mathbf{p}, I) and (\mathbf{q}, J) are stable. Then $I \cdot J = \text{Int}(\mathbf{p}) \cdot \text{Int}(\mathbf{q}) = \text{Int}(\mathbf{p} \cdot \mathbf{q})$. Thus the product is stable, and the stable coloring problems form a Hopf submonoid. Likewise, if $I \cdot J = \text{Int}(\mathbf{p} \cdot \mathbf{q})$, then $I = I \cdot J|_M = \text{Int}(\mathbf{p})$, and similarly $J = I \cdot J|_N = \text{Int}(\mathbf{q})$. Thus, if the product is stable, then so are the factors. Taking the contrapositive, we see that if either factor is unstable, then the product is unstable, and hence the unstable coloring problems form an ideal. Thus, we obtain a combinatorial Hopf monoid. \square

4.3. Linearization and combinatorial Hopf algebras. Recall that a *combinatorial Hopf algebra* is a graded connected Hopf algebra H with a linear character $\varphi : H \rightarrow \mathbb{K}$. In this section we show how to associate a combinatorial Hopf algebra to a combinatorial Hopf monoid.

Given a pointed set species \mathbf{F} , its *linearization* is a functor $\mathbb{K}(\mathbf{F})$ given by defining $\mathbb{K}(\mathbf{F})_N$ to be the vector space on \mathbb{K} with basis given by $\mathbf{F}_N \setminus \{0\}$. Then $\mathbb{K}(\mathbf{F}_M \vee \mathbf{G}_N) \simeq \mathbb{K}(\mathbf{F}_M) \oplus \mathbb{K}(\mathbf{G}_N)$, and $\mathbb{K}(\mathbf{F}_M \wedge \mathbf{G}_N) \simeq \mathbb{K}(\mathbf{F}_M) \otimes \mathbb{K}(\mathbf{G}_N)$. Applying these isomorphisms to pointed set species, we obtain that $\mathbb{K}(\mathbf{F} +$

$\mathbf{G}) \simeq \mathbb{K}(\mathbf{F}) \oplus \mathbb{K}(\mathbf{G})$, and $\mathbb{K}(\mathbf{F} \cdot \mathbf{G}) \simeq \mathbb{K}(\mathbf{F}) \cdot \mathbb{K}(\mathbf{G})$. Stated in terms of Aguiar and Mahajan, we have the following:

Theorem 16. *Linearization is a bistrong monoidal functor. In particular, linearized combinatorial Hopf monoids are Hopf monoids in linear species.*

The proof is given by verifying all the necessary commutative diagrams.

In Aguiar and Mahajan [2], they define a Fock functor, which sends species to graded vector spaces, and sends linear Hopf monoids to graded Hopf algebras. Thus, given a Hopf monoid in pointed set species, we have $\mathcal{F}(\mathbb{K}(\mathbf{H}))$ is a graded Hopf algebra. We give an explicit description of the resulting Hopf algebra. We write $\mathcal{F}(\mathbb{K}(\mathbf{H})) = \bigoplus_{n \geq 0} H_n$, where H_n is the free vector space with basis indexed by isomorphism classes of \mathbf{H} -structures on $[n]$. The multiplication and comultiplication are inherited from the Hopf monoid. That is, we multiply and comultiply basis elements by performing the product and coproduct on representatives of the isomorphism classes. Given an isomorphism class $[h]$, we can define a character by $\varphi([h]) = 0$ if and only if $h \in I$, and 1 otherwise. The fact that I is an ideal of H forces φ to be multiplicative, and thus φ is a character, and $(\mathcal{F}(\mathbb{K}(\mathbf{H})), \varphi)$ is a combinatorial Hopf algebra in the sense of Aguiar, Bergeron, and Sottile [1].

We recall the quasisymmetric function associated to a character on a combinatorial Hopf algebra (H, φ) . Given $h \in H_n$, $\Psi(h) = \sum_{\alpha} \varphi^{\alpha}(h) M_{\alpha}$, where $\varphi^{\alpha} = \varphi^{\otimes |\alpha|} \circ \rho^{\alpha} \circ \Delta^{|\alpha|-1}$, where ρ^{α} is the tensor product of the projection maps from H to H_{α_i} . Note that we are summing over all integer compositions α of n .

Thus, given an element $\mathbf{x} \in \mathbf{H}_I$, where \mathbf{H}_I is a combinatorial Hopf monoid, there is an associated quasisymmetric function $\Psi(h)$. There is a description for Ψ in terms of colorings.

Theorem 17. *Let \mathbf{H} be a combinatorial Hopf monoid. Fix a finite set N , and $\mathbf{h} \in \mathbf{H}_N$. Then*

$$\Psi_{\varphi}(\mathbf{h}) = \sum_f \prod_{n \in N} x_{f(n)}$$

where the sum is over proper colorings.

In the case of posets, we see that a proper coloring of \mathbf{p} is a strict \mathbf{p} -partition, and we recover the classical \mathbf{p} -partition enumerator. In the case of graphs, we recover Stanley's chromatic symmetric function. For coloring problems, we obtain the chromatic quasisymmetric function.

Proof. Given a coloring f , we can define an ordering N by declaring $x \leq y$ if $f(x) \leq f(y)$. This defines a set composition $C = C_1 | C_2 | \cdots | C_k$. If we define a chain S of subsets by $S_i = S_{i-1} \cup C_i$, with $S_0 = \emptyset$, then we see that, for every i , $\mathbf{h}|_{S_i/S_{i-1}}$ is stable. We call this a stable flag. Thus, to each coloring f , we can associate a stable flag $F(f)$. We see that, given a flag F ,

$$\sum_{f: F(f)=F} x_f = M_{\alpha}$$

where $\alpha_i = |S_i \setminus S_{i-1}|$.

Thus, $\sum_f x_f = \sum_\alpha c_\alpha M_\alpha$, where c_α is the number of stable flags of type α . So it suffices to show that the coefficient of M_α in $\Psi_\varphi(h)$ is also c_α . The coproduct of the Hopf algebra is given by $\Delta(h) = \sum_{S \subseteq N} h|_S \otimes h/S$, where we are viewing the elements $h, h|_S$, and h/S up to isomorphism. So $\Delta^{|\alpha|-1}(h) = \sum_{S_1 \subseteq S_2 \subseteq \dots \subseteq S_{|\alpha|}} \bigotimes h|_{S_i/S_{i-1}}$. Then $\varphi^{|\alpha|} \circ \rho^\alpha \circ \Delta^{|\alpha|}(h) = \sum_{S_1 \subseteq S_2 \subseteq \dots \subseteq S_{|\alpha|}} \bigotimes \varphi(h|_{S_i/S_{i-1}})$ where the sum is over all flags where $|S_i \setminus S_{i-1}| = \alpha_i$ for all i . However, $\varphi(h|_{S_i/S_{i-1}})$ is zero, unless $h|_{S_i/S_{i-1}}$ is stable, in which case it is one. The result is c_α . \square

5. FROM HOPF MONOIDS TO COLORING PROBLEMS

In this section, we justify the use of the name ‘coloring problem’. We show that coloring problems for a combinatorial Hopf monoid. Moreover, given a combinatorial Hopf monoid \mathbf{H} , there is a unique morphism $\varphi : \mathbf{H} \rightarrow \mathbb{C}$, and, given $\mathbf{h} \in \mathbf{H}_N$, $\Psi(\mathbf{h}) = \Psi(\varphi(\mathbf{h}))$. In particular, all knowledge of chromatic symmetric functions comes from understanding coloring problems.

Now we discuss the notion of morphism of combinatorial Hopf monoids. Given two species \mathbf{F} and \mathbf{F}' , a morphism is a natural transformation $\varphi : \mathbf{F} \rightarrow \mathbf{F}'$. If \mathbf{F} and \mathbf{F}' are Hopf monoids in species, then the transformation is morphism of Hopf monoids if it preserves products, restrictions, and contractions. That is, $\varphi_S(\mathbf{x}) \cdot \varphi_T(\mathbf{y}) = \varphi_{S \sqcup T}(\mathbf{x} \cdot \mathbf{y})$, $\varphi(\mathbf{x})|_S = \varphi(\mathbf{x}|_S)$, and $\varphi(\mathbf{x})/S = \varphi(\mathbf{x}/S)$. If \mathbf{F} and \mathbf{F}' are combinatorial Hopf monoids, then φ is a morphism of combinatorial Hopf monoids if it preserves products, coproducts, and stable structures. That is, given $\mathbf{f} \in \mathbf{F}_N$, $\varphi(\mathbf{f})$ is stable if and only if \mathbf{f} is stable.

As an example, consider the combinatorial Hopf monoids of posets \mathbf{P} and antimatroids \mathbf{A} . Given a poset \mathbf{p} , there is an associated antimatroid $J(\mathbf{p})$: the antimatroid consists of all ideals of \mathbf{p} . Then J is a morphism of combinatorial Hopf monoids.

Given $\mathbf{x} \in \mathbf{H}_I$, we define $\mathbf{p}(\mathbf{x}) = \{S \subseteq N : x|_S \neq 0\} = \{S \subseteq N : x/S \neq 0\}$. We define $\mathbf{I}(\mathbf{x}) = \{[S, T] : x|_{T/S} \text{ is stable}\}$.

Theorem 18. *Let \mathbf{H} be a combinatorial Hopf monoid. Then there exists a unique morphism $\varphi : \mathbf{H} \rightarrow \mathbb{C}$ of combinatorial Hopf monoids, given by $\varphi(\mathbf{x}) = (\mathbf{p}(\mathbf{x}), \mathbf{I}(\mathbf{x}))$.*

Proof. First, we show that $(\mathbf{p}(\mathbf{x}), \mathbf{I}(\mathbf{x}))$ is a coloring problem. Clearly, $\emptyset, N \in \mathbf{p}(\mathbf{x})$. Moreover, for all $S \subseteq N$, $\mathbf{x}|_S/S = 1$, so $[S, S] \in \mathbf{I}(\mathbf{x})$. So we only need to show that $\mathbf{I}(\mathbf{x})$ is closed under inclusion. Let $[Q, T] \in \mathbf{I}(\mathbf{x})$, and let R, S be such that $Q \subseteq R \subseteq S \subseteq T$. Since $\mathbf{x}|_{T/Q}$ is stable, so is $\mathbf{x}|_{T/Q}|_S = \mathbf{x}|_S/Q$. Likewise, so is $\mathbf{x}|_S/Q/(R \setminus Q) = \mathbf{x}|_S/R$. Thus, $[R, S] \in \mathbf{I}(\mathbf{x})$, and so $\varphi(\mathbf{x})$ is a coloring problem.

φ is multiplicative: Now we show that φ is multiplicative. Let $N = S \sqcup T$, $\mathbf{x} \in \mathbb{H}_S, \mathbf{y} \in \mathbb{H}_T$. Then:

$$\begin{aligned} \mathbf{p}(\mathbf{x} \cdot \mathbf{y}) &= \{M \subseteq N : (\mathbf{x} \cdot \mathbf{y})|_M \neq 0\} \\ &= \{M \subseteq N : \mathbf{x}|_{M \cap S} \cdot \mathbf{y}|_{M \cap T} \neq 0\} \\ &= \{M \subseteq N : \mathbf{x}|_{M \cap S} \neq 0, \mathbf{y}|_{M \cap T} \neq 0\} \\ &= \{M \subseteq N : M \cap S \in \mathbf{p}(\mathbf{x}), M \cap T \in \mathbf{p}(\mathbf{y})\} \\ &= \mathbf{p}(\mathbf{x}) \cdot \mathbf{p}(\mathbf{y}) \end{aligned}$$

where the second equality comes from the fact that restriction is multiplicative, and the third equality comes from the fact that $x \cdot y = 0$ if and only if $x = 0$ or $y = 0$. Similarly,

$$\begin{aligned} \mathbf{I}(\mathbf{x} \cdot \mathbf{y}) &= \{[L, M] : (\mathbf{x} \cdot \mathbf{y})|_M / L \text{ is stable}\} \\ &= \{[L, M] : (\mathbf{x}|_{M \cap S} / (L \cap S)) \cdot (\mathbf{y}|_{M \cap T} / (L \cap T)) \text{ is stable}\} \\ &= \{[L, M] : \mathbf{x}|_{M \cap S} / (L \cap S) \text{ and } \mathbf{y}|_{M \cap T} / (L \cap T) \text{ are both stable}\} \\ &= \{[L, M] : [L \cap S, M \cap S] \in \mathbf{I}(\mathbf{x}), [L \cap T, M \cap T] \in \mathbf{I}(\mathbf{y})\} \\ &= \mathbf{I}(\mathbf{x}) \cdot \mathbf{I}(\mathbf{y}) \end{aligned}$$

where the second equality also uses multiplicativity of restriction and contraction, and the third equality uses the fact that $x \cdot y$ is stable if and only if both x and y are stable.

φ preserves restriction: Now we show that φ preserves restrictions. Let $\mathbf{x} \in \mathbb{H}_N$, and $S \subseteq N$. If $\mathbf{x}|_S = 0$, then $\mathbf{p}(\mathbf{x}|_S) = 0$. Since $\mathbf{x}|_S = 0$, $S \notin \mathbf{p}(\mathbf{x})$, so $\mathbf{p}(\mathbf{x})|_S = 0$, so we have equality. Assume then that $\mathbf{x}|_S \neq 0$. Then

$$\begin{aligned} \mathbf{p}(\mathbf{x}|_S) &= \{R \subseteq S : (\mathbf{x}|_S)|_R \neq 0\} \\ &= \{R \subseteq S : \mathbf{x}|_R \neq 0\} \\ &= \{R \subseteq S : R \in \mathbf{p}(\mathbf{x})\} \\ &= \mathbf{p}(\mathbf{x})|_S. \end{aligned}$$

Similarly,

$$\begin{aligned} \mathbf{I}(\mathbf{x}|_S) &= \{[Q, R] : R \subseteq S, (\mathbf{x}|_S)|_R / Q \text{ is stable}\} \\ &= \{[Q, R] : R \subseteq S, \mathbf{x}|_R / Q \text{ is stable}\} \\ &= \{[Q, R] : R \subseteq S, [Q, R] \in \mathbf{I}(\mathbf{x})\} \\ &= \mathbf{I}(\mathbf{x})|_S. \end{aligned}$$

φ preserves contraction: We show that $\varphi(\mathbf{x}/S) = \varphi(\mathbf{x})/S$. First,

$$\begin{aligned} \mathbf{p}(\mathbf{x}/S) &= \{T \subseteq N \setminus S : (\mathbf{x}/S)/T \neq 0\} \\ &= \{T \subseteq N \setminus S : \mathbf{x}/(T \sqcup S) \neq 0\} \\ &= \{T \subseteq N \setminus S : T \sqcup S \in \mathbf{p}(\mathbf{x})\} \\ &= \mathbf{p}(\mathbf{x})/S. \end{aligned}$$

By a similar calculation,

$$\begin{aligned} \mathbf{I}(\mathbf{x}/S) &= \{[Q, R] : R \subseteq N \setminus S, (\mathbf{x}/S)|_R / Q \text{ is stable}\} \\ &= \{[Q, R] : R \subseteq N \setminus S, \mathbf{x}|_{R \sqcup S} / S / Q \text{ is stable}\} \\ &= \{[Q, R] : R \subseteq N \setminus S, \mathbf{x}|_{R \sqcup S} / (Q \sqcup S) \text{ is stable}\} \\ &= \{[Q, R] : R \subseteq N \setminus S, [Q \sqcup S, R \sqcup S] \in \mathbf{I}(\mathbf{x})\} \\ &= \mathbf{I}(\mathbf{x})/S. \end{aligned}$$

Uniqueness of φ : We claim that φ is unique. Let $\psi : \mathbf{H} \rightarrow \mathbf{C}$ be another morphism of combinatorial Hopf monoids, and let $\mathbf{h} \in \mathbf{H}_N$. Let (\mathbf{q}, \mathbf{J}) be the coloring problem $\psi(\mathbf{h})$. Let $S \in \mathbf{p}(\mathbf{h})$. Then $\mathbf{h}|_S \neq 0$, and $\mathbf{h}/S \neq 0$. Since $\psi(\mathbf{h})|_S = \psi(\mathbf{h}|_S) \neq 0$, and $\psi(\mathbf{h})/S = \psi(\mathbf{h}/S) \neq 0$, it follows that $S \in \mathbf{q}$. Likewise, if $S \notin \mathbf{p}$, then $\mathbf{h}|_S = 0$. Hence $\psi(\mathbf{h})|_S = \psi(\mathbf{h}|_S) = 0$, and hence $S \notin \mathbf{q}$. Therefore $\mathbf{p} = \mathbf{q}$.

Let $[S, T] \in \mathbf{I}(\mathbf{h})$. Then $\mathbf{h}|_{T/S}$ is stable, and hence $\varphi(\mathbf{h}|_{T/S}) = \varphi(\mathbf{h})|_{T/S}$ is also stable. However, by definition of restriction and contraction for coloring problems, this means that $\mathbf{q}|_{T/S}$ is stable, so $\mathbf{J}|_{T/S} = \text{Int}(\mathbf{q}|_{T/S})$. In particular, $[\emptyset, T \setminus S] \in \mathbf{J}|_{T/S}$, so $[S, T] \in \mathbf{J}$.

Let $[S, T] \in \mathbf{J}$. Then $(\mathbf{q}, \mathbf{J})|_{T/S}$ is stable, so $\varphi(\mathbf{h})|_{T/S} = \varphi(\mathbf{h}|_{T/S})$ is too. However, this means that $\mathbf{h}|_{T/S}$ is stable, and hence $[S, T] \in \mathbf{I}$. Therefore $\mathbf{I} = \mathbf{J}$. Thus $\varphi(\mathbf{h}) = \psi(\mathbf{h})$, and uniqueness follows. \square

We remark that the definition of combinatorial comonoid guarantees that the map φ is a morphism of comonoids. It is possible to work with a definition of combinatorial Hopf monoid that does not impose the combinatorial comonoid restrictions. It is also possible to construct a terminal object in the resulting category. However, the resulting object is very difficult to describe: they appear in [11], as ‘composition complexes’. Likewise, if we no longer require that stable sets form a subcomonoid, then the terminal object consists of collections of set compositions.

In summary, the axioms chosen for defining combinatorial Hopf monoids leads to a natural combinatorial object, namely coloring problems. Moreover, coloring problems are intimately connected with geometry, leading to the possibility of using geometric tools to study chromatic polynomials. Finally, there are lots of examples of combinatorial Hopf monoids: the primary examples of bimonoids in pointed set species that are not combinatorial Hopf monoids, in our sense, are precisely the three examples given in the previous paragraphs. In particular, it seems worthwhile to continue with the more restrictive definition.

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